

Angular momentum & torque



Angular momentum (L)

It is defined as the moment of linear momentum of a body.

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}$$

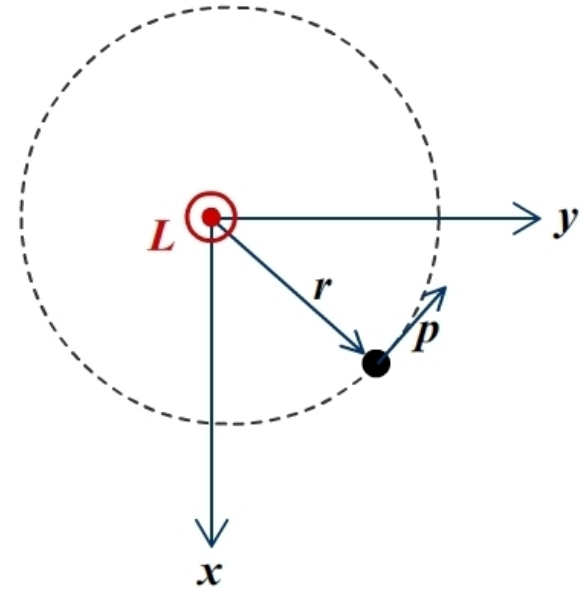
$$\mathbf{L} = r \times p \sin(\theta) \hat{n}$$

θ is the angle between $\mathbf{r}$ and $\mathbf{p}$.

$\hat{n}$ is a unit vector whose direction is given by the right hand thumb rule.

$\mathbf{L}$ is a vector quantity

Its SI unit is $\text{kg m}^2 \text{s}^{-1}$



Other forms of above relation

(considering only the magnitude and $\theta = 90^\circ$)

$$L = m v r$$

$$L = m r^2 \omega$$

$$L = I \omega$$

Torque (τ)

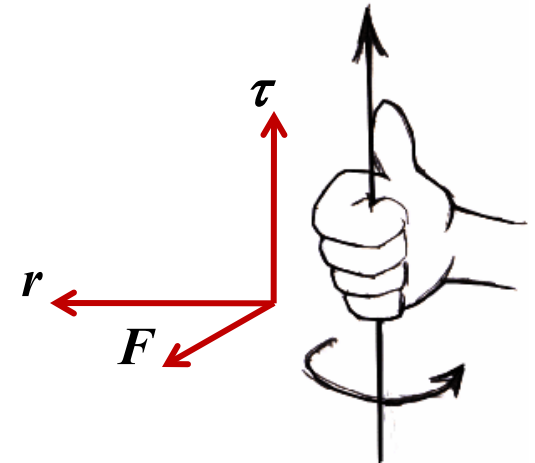
It is defined as the moment of force acting on a body.

$$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$$

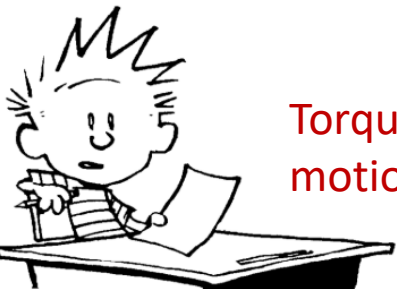
$$\boldsymbol{\tau} = |\mathbf{r}| |\mathbf{F}| \sin(\theta) \hat{n}$$

It is defined as the moment of force acting on a body.

- Torque is a vector quantity
- Direction of torque is obtained using right hand thumb rule
- SI unit : Nm



Torque, in rotatory motion, plays the same role as force in translatory motion. You will see this similarity in many situations.



Relation between torque and angular acceleration

Torque acting on a body is related to angular momentum as

$$\tau = \frac{dL}{dt}$$

Using the relation $L = I \omega$, we get

$$\tau = \frac{dI\omega}{dt}$$

Assuming the moment of inertia of the body to be constant

$$\tau = I \frac{d\omega}{dt}$$

$d\omega/dt$ is by definition angular acceleration (α). Therefore

$$\tau = I \alpha$$

This derivation approach is very similar to the one used to derive $F = ma$ using Newton's second law of motion.



Relation between torque and angular acceleration (alternate approach)

Force is related to acceleration in the body as

$$\mathbf{F} = m\mathbf{a}$$

Taking cross product with $\mathbf{r}$ on both sides we get

$$\mathbf{r} \times \mathbf{F} = m(\mathbf{r} \times \mathbf{a})$$

Using the relations $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ and $\mathbf{a} = \boldsymbol{\omega} \times \mathbf{r}$ we get

$$\boldsymbol{\tau} = m\mathbf{r} \times (\boldsymbol{\alpha} \times \mathbf{r})$$

$$\tau = m r^2 \alpha$$

mr^2 is the moment of inertia (I) of the body, Therefore

$$\tau = I \alpha$$

In the derivation we have used vector triple product i.e.

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \mathbf{B}(\mathbf{A} \cdot \mathbf{C}) - \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$$



Law of conservation of angular momentum

If the net torque acting on a system is zero, then the total angular momentum of a system remains constant.

$$\tau = \frac{dL}{dt}$$

If the net torque acting on a system is zero, we get

$$\frac{dL}{dt} = 0$$

$$\Rightarrow L = \text{constant}$$

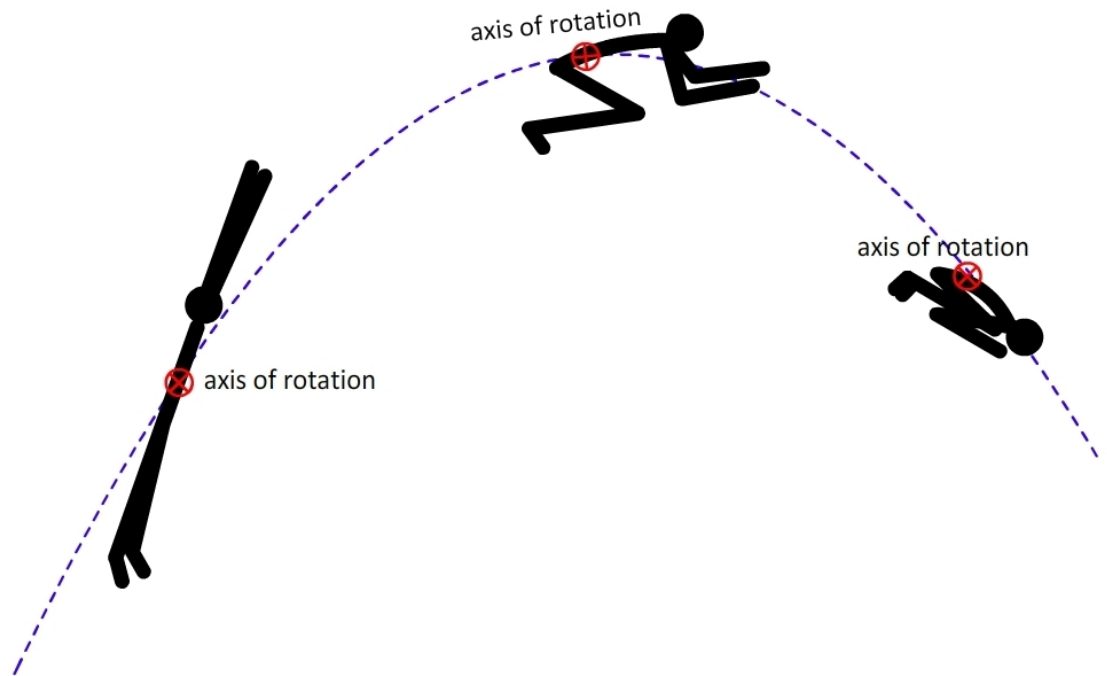
Examples of law of conservation of angular momentum

Platform diving : A person jumps off a platform from a certain height with his arms and legs stretched outwards. Let his initial angular velocity and moment of inertia be denoted as ω_1 and I_1 respectively. In mid air he curls himself inwards there by reducing his moment of inertia. Let his initial angular velocity and moment of inertia in this state be denoted as ω_2 and I_2 respectively. As the torque acting on him is zero, his angular momentum should be constant. Therefore

$$I_1\omega_1 = I_2\omega_2$$

$$\omega_2 = \left(\frac{I_1}{I_2} \right) \omega_1$$

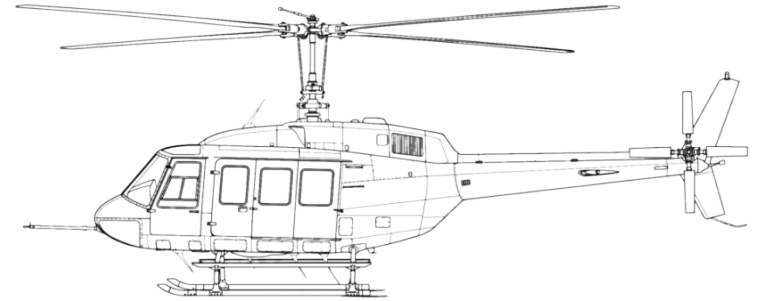
As I_2 is less than I_1 , he is able to execute a few summersaults before stretching himself outwards again and dive into the pool.



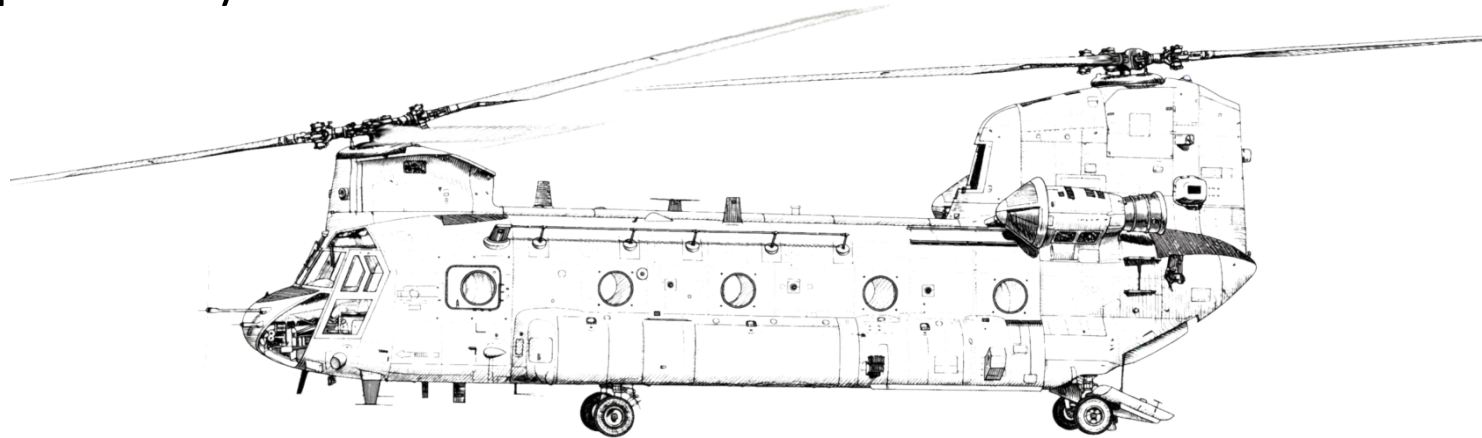
Examples of law of conservation of angular momentum

Two blades of a helicopter :

A helicopter should have two blades. As the helicopter lifts off the ground, the change in angular momentum caused by the main rotor is compensated by the tail rotor as it generates torque by pushing the air at the far end.



In a helicopter with two main rotors, the blades rotate in opposite directions with the angular momentum of one balancing the angular momentum of the other. This keeps the helicopter steady and stable.



Examples of law of conservation of angular momentum

It is easier to balance a bicycle in motion

It is easier to balance a bicycle in motion than at rest. This is because of conservation of angular momentum resulting from gyroscopic effect.

Due to gyroscopic effect a spinning (rotating) object tends to oppose any change in its angular momentum arising out of change in its axis of rotation.



The *slow bicycle race* is a unique race in which the last one to cross the finish line is the winner. This race tests the balancing skills of the rider as the gyroscopic effect in such a race is negligible !

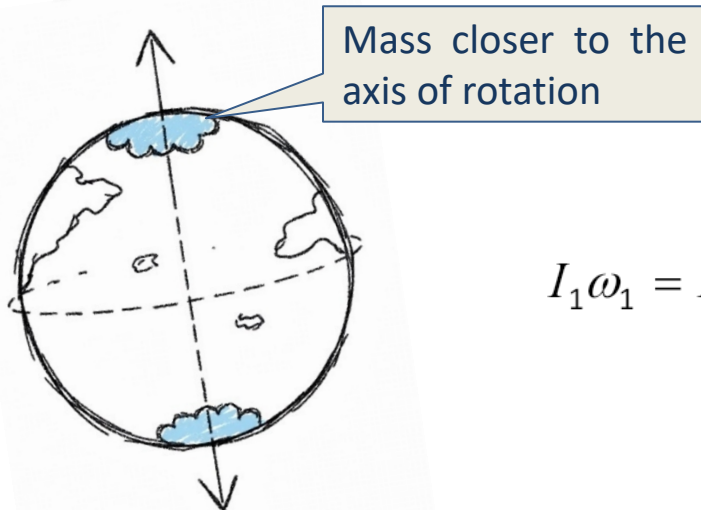
Examples of law of conservation of angular momentum

What if polar ice caps melt?

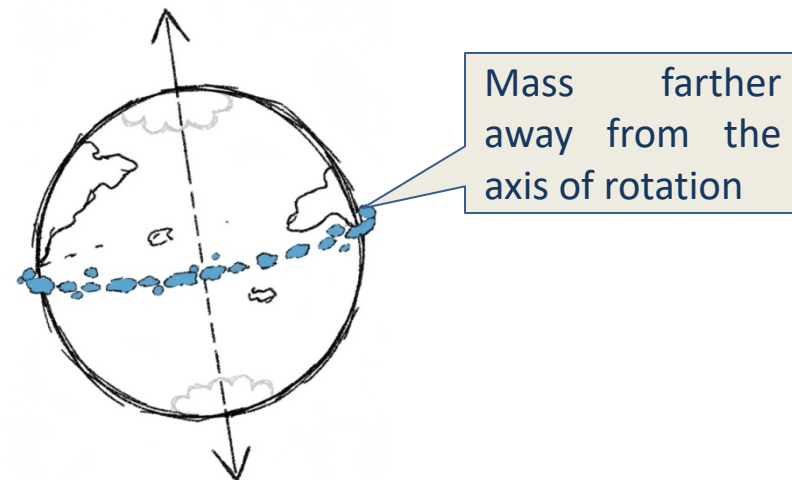
Present duration of the day is due to the angular velocity of the earth and its moment of inertia about its axis of rotation.

Net torque acting on the earth is zero and therefore its angular momentum ($I\omega$) is constant.

If the polar ice caps melt and the water flows towards the equator, the moment of inertia of the earth increases. As the angular momentum is constant, this results in a decrease in its angular velocity and increase in the duration (time taken for one complete rotation).



$$I_1\omega_1 = I_2\omega_2$$



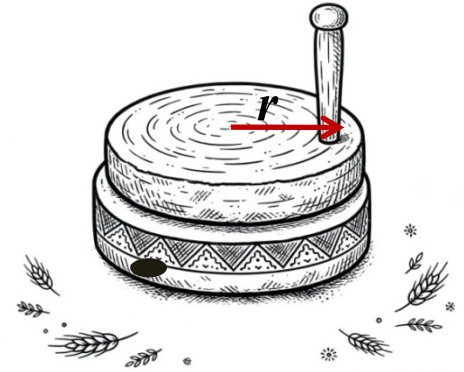
Examples of application of torque on a body

Traditional hand flour grinder

Torque required to rotate the flour grinder is given by

$$\tau = r \times F$$

The handle of the grinder is placed near its edge. This increase the value of r . This results in lesser force required to cause rotation in the grinder.

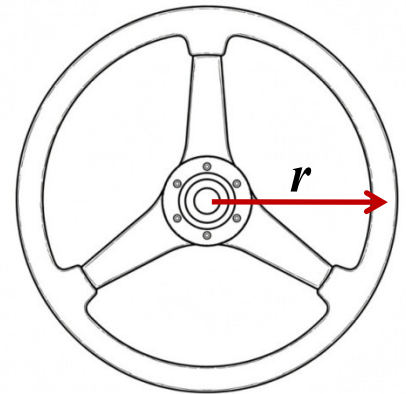


Steering wheel of a car

Torque required to rotate the steering wheel is given by

$$\tau = r \times F$$

For a steering wheel of large diameter, the value of r increases. This results in lesser force required to cause rotation in the steering wheel.



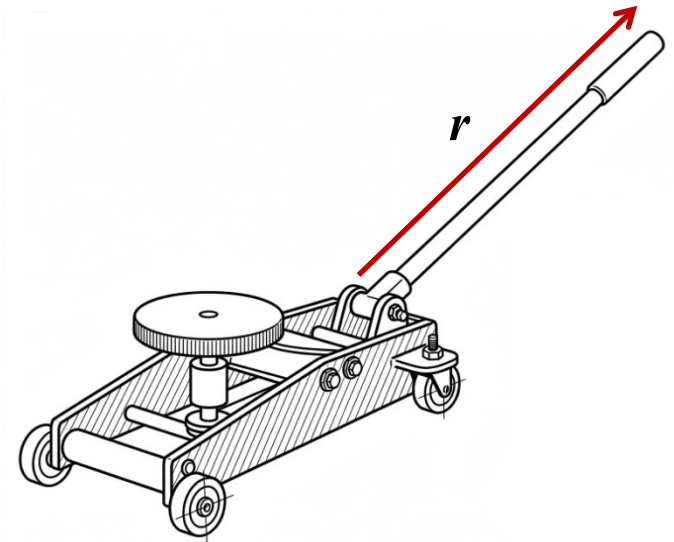
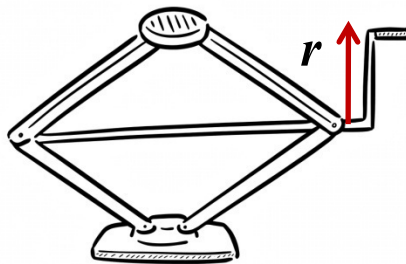
Examples of application of torque on a body

Jack screw is provided with a long handle

Torque required to operate the jack screw is given by

$$\tau = r \times F$$

With a long handle, the value of r increases. This results in lesser force required to operate the jack screw.



System of particles and rotation

Examples of application of torque on a body

We cannot open / close doors by applying force at the hinges

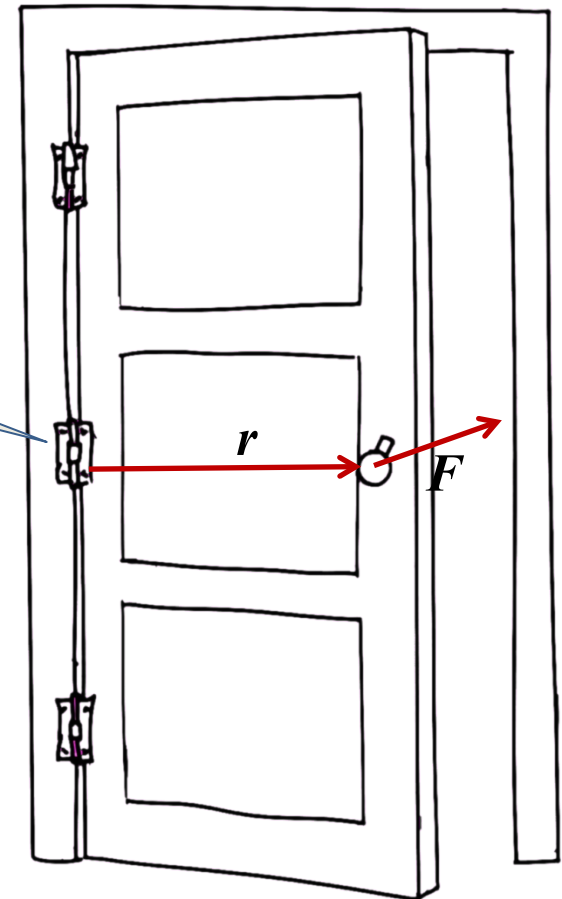
Torque required to open / close a door is given by

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$

When the force is applied at the hinges, the value of $\boldsymbol{r}$ becomes zero. This results in zero torque !

Therefore we cannot open / close a door by applying force at the hinges.

At the hinges $\boldsymbol{r}$
becomes zero !!

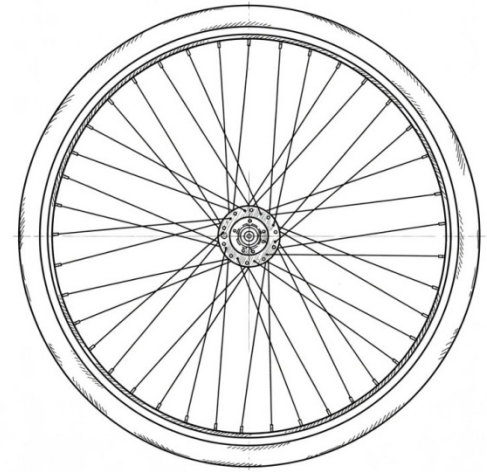


Additional applications

Use of spokes in bicycles

To increase moment of inertia

Spokes increase the moment of inertia of the wheel without contributing to the mass. A wheel with spokes is much lighter than a solid disc wheel, but equally strong. This helps in easy acceleration and handling.



To support and distribute Load: Another main function of spokes is to support the rim and distribute the weight of the rider and the bicycle evenly around the wheel. They work on the principle of tension. The hub hangs from the upper spokes.

To Keep the Wheel True: Spokes are tightened to a precise tension to keep the wheel perfectly round and straight. This is called keeping the wheel "true" and it prevents wobbling while riding.

To Transmit Forces: Spokes transmit mechanical forces. While pedaling, the torque from the hub is transferred to the rim through the spokes. Similarly, during braking, the force is transferred from the rim to the hub.

To Act as Shock Absorber: The slight elasticity of spokes absorbs small road shocks and gives a comfortable ride.

System of particles and rotation

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